Determination of the Finite Quaternion Groups.

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If we apply to a set of quaternions the definition of a group as given in the Theory of Substitutions, then a *quaternion group* of the m^{th} order means a set of m quaternions (scalar unity always included) whose products and powers are also quaternions of the same set or group. The object of the present paper is to determine all the possible finite quaternion groups.

These groups, or rather their analogues in the ordinary Theory of Functions, have usually been interpreted geometrically as the linear transformations of the plane of complex variables in itself—*automorphic transformations*. The formulae for these linear transformations were first given by Professor Gordan in his paper "Ueber endliche Gruppen linearer Transformationen einer Veränderlichen."* The quaternion formulae, although they have their exact correspondents in Gordan's algebraic formulae, have, when interpreted geometrically, a more general character, in that they represent certain automorphic linear transformations of a three-dimensional infinite homoloidal space, or what is the same thing, of a three-fold extended sphere. The determination of these formulae might be made to depend upon Gordan's solution, but the following is a shorter and simpler solution than that of Gordan, and includes it as a special case.

Some General Formulae.

Let q, r represent two quaternions belonging to a finite group. In order that the group may be finite, the tensors of q, r, etc., must be unity, so that q and r may be written in the form

$$q = \cos\phi + \lambda \sin\phi,$$
$$r = \cos\psi + \mu \sin\psi,$$

* *Math. Ann.* Bd. XII, pp. 23–46. Compare also Cayley "On the finite Groups of linear Transformations of a Variable," *Math. Ann.* Bd. XVI, pp. 260–63, 439–40. The proof concerning the number and character of the possible groups was given by Prof. Klein, *Math. Ann.* Bd. IX, pp. 183–208: "Ueber binäre Formen mit linearen Transformationen in sich selbst."

ϕ, ψ being the arguments, λ, μ the axes of q, r respectively. According to definition qr is also a member of the group, and if we write

(i) $$qr = \cos\chi + \nu \sin\chi,$$

χ represents a new argument and ν a new axis of the group, and they are determined by the relations

(ii) $$\nu \sin\chi = \lambda \sin\phi \cos\psi + \mu \sin\psi \cos\phi + V\lambda\mu \sin\phi \sin\psi,$$

(iii) $$\cos\chi = \cos\phi \cos\psi + S\lambda\mu \sin\phi \sin\psi$$

$$= \cos^2\frac{f}{2} \cos(\phi+\psi) + \sin^2\frac{f}{2} \cos(\phi-\psi),$$

where f is the angle formed by λ with μ.

In particular, we have the well-known formulae

$$q^2 = \cos 2\phi + \lambda \sin 2\phi$$

$$\cdots\cdots\cdots\cdots$$

(iv) $$q^n = \cos n\phi + \lambda \sin n\phi.$$

In order that q may belong to a finite group, it is further necessary that some finite power of q should reproduce the *identity*—*i. e.* unity—so that, *e. g.*, $q^m = 1$; ϕ is then a rational multiple of ∂; m is called the period of q.

I. In the following discussion it is important to remember, that *when two quaternions of the same group have the same axis, they can always be written as powers of one and the same quaternion.*

Limits of the Periods.

Amongst the quaternions of the group there is one set—or one quaternion at least—whose argument is the smallest which the group contains. Let q be such a quaternion, and moreover such that its axis λ forms with some other axis μ of the group the smallest angle (the case $\lambda = \pm\mu$ excluded) which the axis of *any* quaternion of the group with the smallest argument can form with *any* other axis of the group. Under this hypothesis we have, if r be the quanternion whose axis is μ,

$$TV.\mu UVrqr \geqq TV\lambda\mu.$$

From the product rqr is deduced easily

$$V.\mu Vrqr = TVq \,.\, V\mu\lambda,$$

(v) $$V.\mu UVrqr = \frac{TVq}{TVrqr} \cdot V\mu\lambda.$$

In order that the condition $TV.\mu UVrqr \geqq TV\lambda\mu$ may be fulfilled, we must therefore have

(vi) $$V^2 q \geqq V^2 rqr\,, \text{ or } S^2 q \leqq S^2 qr^2\,,$$

i. e. $$\cos^2\phi \leqq \cos^2\theta\,,$$

where θ is the argument of rqr; ϕ, however, is by hypothesis the smallest argument which the group can possess, so that the only possible supposition is

$$\cos^2\phi = \cos^2\theta\,, \text{ or } Sqr^2 = \pm Sq\,.$$

Remembering that $S^2 r - V^2 r = 1$ (since $Tr = 1$) and that $r = Sr + Vr$, we have, if the sign before the second member be plus,

(vii) $$Sr\,.\,Sqr = 0\,,$$

but if minus,

$$S.rqr\,Vr = 0\,.$$

The latter of these conditions, however, would give

$$S\lambda\mu\,.\,\tan\phi = \tan\psi\,;$$

now as ψ by hypothesis cannot be smaller than ϕ, we should have

$$S\lambda\mu = \pm 1\,, \; i.\,e.\; \lambda = \pm\mu\,,$$

and r would be a power of q, a case we evidently do not need to consider. We have remaining therefore only the former condition $Sr\,.\,Sqr = 0$, which compels us to write either

$$Sr = 0\,, \text{ or } Sqr = 0\,.$$

The case $Sr = 0$ affords a possible solution, and determines for r the period 4.

But perhaps $Sqr = 0$ is also a solution. On the supposition that it is, write qr in the form

$$qr = q^{\frac{1}{2}} q^{\frac{1}{2}} r q^{\frac{1}{2}} q^{-\frac{1}{2}}\,.$$

This equation shows that the axes of qr and $q^{\frac{1}{2}} r q^{\frac{1}{2}}$ make equal angles with the axis of q. This angle cannot be smaller than the angle between λ and μ, and we have, if we rewrite the condition (vi) for this case,

(vi)′ $$S^2 qr \geqq S^2 r\,,$$

so that the condition $Sqr = 0$ demands also that $Sr = 0$; that is, the two conditions in question are equivalent, and r has in any case the period 4. We have therefore the following result:

II. *If q and r belong to a finite quaternion group, and if the argument of q is the smallest which the group contains, and if the angle between the axes of q and r is the smallest one existing between the axis of a quaternion with the smallest argument*

and any other axis of the group (the angle zero or $2\circlearrowleft$ *excepted), then* $Sr=0$ *and* r *has of necessity the period* 4.

One quaternion belonging to the group is

$$q' = rqr^{-1} = \cos\phi + \lambda' \sin\phi,$$

where $\lambda' = \mu\lambda\mu^{-1}$. If now we form the product $qq'q$—which must also be a quaternion of the group—we shall find, by the same process as that by which (v) was found, that

$$\text{(v)}' \qquad V.\lambda U Vqq'q = \frac{\sin\varphi}{TVqq'q} \cdot V\lambda\lambda',$$

and since ϕ is the smallest argument of the group,

$$\sin\phi \leqq TVqq'q,$$

from which and the preceding equation it follows, that

$$\text{(viii)} \qquad TV.\lambda UVqq'q \leqq TV\lambda\lambda'.$$

This last inequality shows that the axis of $qq'q$, if it do not coincide with λ or λ', must lie between them and therefore coincide with μ, for μ is the only axis lying between these limits. We have therefore three possibilities:

$$qq'q = q'^x, \text{ or } qq'q = q^y, \text{ or } qq'q = \mu^z,$$

where x, y, z are integers. The case $qq'q = q'^x$, if we develop the equation $qq' = q'^x q^{-1}$ and operate with $V.\lambda$ on the result, gives

$$0 = V.\lambda\lambda' \operatorname{ctn}\phi + V.\lambda V\lambda\lambda',$$

$$\text{(ix)} \qquad 0 = V.\lambda\lambda' \operatorname{ctn}\phi - \lambda' - \lambda S\lambda\lambda';$$

and finally, by means of the operation $S.\lambda'$,

$$S^2\lambda\lambda' = 1,$$

which means simply—according to (I)—that q' is a power of q. The second case $qq'q = q^y$ and the case $qq'q = \pm\mu^2 = \pm 1$, give a similar result, since q' becomes respectively $q' = q^{y-2}$ and $q' = \pm q^{-2}$. We therefore conclude,

III. *That the quaternion* $q\mu q\mu^{-1}q$, *if it belong to a finite group, is either a power of* q, *or is equal to* $\pm\mu$; *provided* μ, *as compared with all the other axes of the group, forms with the axis of* q *the minimum angle.*

Groups which have a period greater than 10.

In order that the equation $qq'q = q^y$ (or $q' = q^{y-2}$) may be satisfied, we must have $\mu\lambda\mu^{-1} = \pm\lambda$, which demands further, that either

$$\text{(x)} \qquad V\lambda\mu = 0, \text{ or}$$

$$\text{(xi)} \qquad S\lambda\mu = 0,$$

that is, μ is either parallel or perpendicular to λ. If however $qq'q = \pm\mu$ should be satisfied, then

$$Sqq'q = \cos\phi\cos 2\phi + S\lambda\lambda'\sin\phi\,\sin 2\phi = 0,$$

(xii) $$S\lambda\lambda' = -\operatorname{ctn}\phi\operatorname{ctn}2\phi.$$

The smallest value which ϕ can here assume is $\frac{\partial}{6}$, and corresponding to this value we have

$$S\lambda\lambda' = -1, \qquad \lambda' = \lambda = \mu = q^3;$$

that is, r is a power of q, a case we do not consider. The condition $qq'q = \pm\mu$, therefore, is only possible when ϕ is greater than $\frac{\partial}{6}$, *i. e.* when q has a period less than 12; and since it will be shown in the sequel (p. 353, VI) that the largest period of a finite group having more than a single axis must be an even period, it follows that the condition in question can exist only when the period of q does not exceed 10. We have, therefore, if the period of q be greater than 10, the single condition (xi), the condition (x) being irrelevant. Since of all the axes of the group μ has the greatest inclination to the axis of q, it follows from the results just obtained, that

IV. *If q has a period greater than* 10, *all the axes of the group, except that of q itself, lie in the plane—passing through the origin—to which the axis of q is perpendicular.*

It will next be shown, that

V. *Of a group containing a period greater than* 10, *all the quaternions which are not powers of q have either the period* 4 *or* 2.

If there be a quaternion s of the group, whose period is different from 4 or 2 and which is not a power of q, then there must be several such—with different axes, but with the same argument—which can be formed by means of the transformations qsq^{-1}, q^2sq^{-2}, . . . Let us suppose, therefore, that there are two such quaternions s, s' with the argument χ and the axes ν, ν'. Then by IV, either Vss' coincides with λ, or else it lies in the same plane with Vs and Vs'; that is, we must have either

$$S.\nu Vss' = S.\nu' Vss' = 0,$$

or $$S.\nu\nu' Vss' = 0.$$

These two equations are equivalent respectively to

$$(S\nu\nu' - 1).\sin 2\chi = 0$$

and $$V^2\nu\nu'.\sin^2\chi = 0.$$

Either equation would be satisfied by $S\nu\nu' = 1$, or what is the same thing, by $V\nu\nu' = 0$, in which case, however, ν and ν' would, contrary to hypothesis, coincide; hence χ must be so chosen as to make either sin 2χ or sin χ vanish, that is to say, χ must be an integral multiple either of $\eth$, or of $\frac{\eth}{2}$. — $Q.E.D.$

Cyclotomic (Kreistheilungs) Groups. If the condition $qq'q = q^y$, *i. e.* $V\lambda\mu = 0$, be fulfilled, all the axes of the group coincide with λ.

The group contains then only powers of a single quaternion q. If q be written in the form

$$q = \cos\frac{2\eth}{m} + \lambda\sin\frac{2\eth}{m},$$

we may assume as the generator of the group any power of q of the form q^a, where a is prime to m; the powers of q^a form a cyclotomic group of the m^{th} order. Since there are $\phi(m)$ numbers prime to m, there are $\phi(m)$ such cyclotomic groups composed of powers of q. Regarded as a whole, such groups are repetitions of each other written in different order.

If, besides the powers of q, there exist, as belonging to the group, only quaternions of the period 2, such quaternions are evidently all identical with each other and equal to -1. If this be combined with all the quaternions of a cyclotomic group of even order, say m, the resulting group will be a cyclotomic group of order $2m$, composed of the powers of $(-q)$. If, on the other hand, a cyclotomic group of even order be combined with -1, the result will be the same cyclotomic group again.

The Double-Pyramid Groups. If besides the m powers of q, m being even, there exists in the group a quaternion having the period 4, which we may represent by the vector μ—and μ is in this case also perpendicular to λ— then, by combining μ with all the powers of q, we can form m other quaternions

$$\mu q,\ \mu q^2,\ \ldots\ \mu q^m,$$

(or what is the same thing $q\mu$, $q^2\mu$, ... $q^m\mu$, since $\mu q^x = q^{-x}\mu$) which together with the m powers of q compose a group of the order $2m$. If, however, m be odd, then besides the $2m$ quaternions just mentioned, the group will contain $2m$ other quaternions

$$\begin{array}{ll} \mu^2 q,\ \mu^2 q^2,\ \ldots\ \mu^2 q^m & [\mu^2 = -1] \\ \mu^3 q,\ \mu^3 q^2,\ \ldots\ \mu^3 q^m & [\mu^3 = -\mu], \end{array}$$

and the group will therefore be of the order $4m$. Hence the order of every quaternion group of this type is a multiple of 4.

Suppose $4m$ to be the order of a double-pyramid group. Then q has the period $2m$, and we have just seen that there are $\phi\,(2m)$ different powers of q, represented by q^a (where a is prime to m) any one of which will generate the cyclotomic group of the $2m^{\text{th}}$ order. We can use any one of these powers q^a as one of the generators of the double-pyramid group of the $2m^{\text{th}}$ order, μ being the other generator, and can thus obtain this group also in $\phi\,(2m)$ different forms.

Groups containing no Period greater than 10.

The Period 10. If q has a period not exceeding 10, we have then, for the determination of the angle between λ and μ, the condition (xii), p. 349, viz:

(xii) $$S\,.\,(\lambda\mu)^2 = S\lambda\lambda' = -\operatorname{ctn}\phi\,\operatorname{ctn}2\phi.$$

For the period 10, therefore, we have

$$S\lambda\lambda' = -\cos 2f = -\operatorname{ctn}\frac{\partial}{5}\cdot\operatorname{ctn}\frac{2\partial}{5},$$

if $2f$ denote the angle which λ makes with λ', *i. e.* if f be the angle between λ and μ. Hence

$$\cos 2f = \frac{1}{\sqrt{5}},\quad \cos f = \sqrt{\frac{e}{\sqrt{5}}} = -S\lambda\mu$$

where $e = \dfrac{1+\sqrt{5}}{2}$. If we write $\mu = i$ and make the plane of $(\lambda,\ \mu)$ that of $(i,\ k)$, we shall find the values of λ and q to be

$$\lambda = i\cos f + k\sin f = \frac{i - e'k}{\sqrt{1+e'^2}},$$

$$q = \cos\frac{\partial}{5} + \lambda\sin\frac{\partial}{5} = \frac{e + i - e'k}{2},$$

where $e' = \dfrac{1-\sqrt{5}}{2}$. By combining q and i and their powers and products in all possible ways, we shall generate a closed group consisting of 120 different quaternions:—the *Double-Ikosahedron Group.* q and i are called the generators of the group. (The quaternions of this and the following groups are written out in full in the accompanying tables $I_m - Q_{120}$, pp. 354, 355.)

Having chosen $\mu = i$, we might then have written, instead of

$$\lambda = i\cos f + k\sin f,$$

this value:

$$\lambda = i\cos f + j\sin f.$$

The latter assumption would evidently—on account of the symmetry of the $i,\ j,\ k$, in their relations to each other—result in a group similar to the

one determined by the former value of λ. It will be found that the two groups differ only by an interchange of e and e', and it is also easy to show that the former group becomes the latter by applying to it the transformation $(1-k)(\ )(1-k)^{-1}$; that is, by writing, in the former group, for each quaternion q the quaternion $(1-k).q.(1-k)^{-1}$. Two such groups are called similar and isomorphic*.

The Period 8. If q has the period 8, then $\phi = \frac{\partial}{4}$ and

$$S\lambda\lambda' = -\cos 2f = -\operatorname{ctn}\frac{\partial}{4}\operatorname{ctn}\frac{\partial}{2} = 0,$$

$$f = \frac{\partial}{4}.$$

Writing $\lambda = i$, we have

$$q = \frac{1+i}{\sqrt{2}},$$

and since μ forms with λ an angle of 45°, we may assume

$$\mu = \frac{i+j}{\sqrt{2}}.$$

q and μ will then be the generators of a closed group of 48 quaternions:—the *Double-Oktahedron Group.* The choice $\mu = \frac{i+k}{\sqrt{2}}$ could evidently equally well have been made; it turns out however that q and $\frac{i+k}{\sqrt{2}}$ generate the same group as q and $\frac{i+j}{\sqrt{2}}$.

The Period 6. If 6 be the period of q, then $\phi = \frac{\partial}{3}$ and

$$S\lambda\lambda' = -\cos 2f = -\operatorname{ctn}\frac{\partial}{3}\operatorname{ctn}\frac{2\partial}{3} = \frac{1}{3},$$

$$\cos f = \frac{1}{\sqrt{3}}.$$

Here, if we write $\mu = i$ and determine the plane (λ, μ) in such a way that it bisects the angle between j and k, we shall have for the values of λ and q

$$\lambda = \frac{j+k}{\sqrt{2}}\sin f + i\cos f = \frac{i+j+k}{\sqrt{3}},$$

$$q = \frac{1+i+j+k}{2}.$$

* C. Jordan, Traité des substitutions et des equations algebriques, pp. 23, 56.

q and i are the generators of a group of the 24^{th} order:—the *Double-Tetrahedron Group.*

The Period 4. If q have the period 4, then $q = \lambda$. If the condition (xii) be applied to this case, we find $Sqq'q = 0 = S\lambda\lambda'$, $f = \frac{\partial}{4}$, and thus obtain the same value of f which gave rise to the double-oktahedron group; and we should in fact obtain this same group for the above value of f in the present case. This results from the fact that the product $\lambda\mu$ would be a quaternion having 8 for its period. We may therefore throw aside the condition (xii) and applying (xi)—*i. e.* $S\lambda\mu = 0$—write

$$\lambda = i, \qquad \mu = j,$$

which two quaternions are the generators of the well-known group of the 8^{th} order, viz. ± 1, $\pm i$, $\pm j$, $\pm k$. This is a special example of the double-pyramid type.

The Periods 11, 9, 7, 5, 3. In order to prove that the above enumeration of the finite quaternion groups is exhaustive, it will only be necessary to show,

VI. *That the greatest period contained in any finite quaternion group, which is not a cyclotomic group, is even.*

The smallest argument which can belong to an odd period m is $\frac{2\partial}{m}$. If such an argument however belong to a finite group which possesses more than a single axis, it cannot be the smallest argument in the group; for if q be the quaternion whose argument is $\frac{2\partial}{m}$, m being odd, and if μ be as before the axis most inclined to that of q, then $qm^2q^{\frac{m-1}{2}} = -q^{\frac{m+1}{2}}$ belongs to the group, and since

$$\begin{aligned} -q^{\frac{m+1}{2}} &= -\cos\frac{m+1}{2}\partial - \lambda\sin\frac{m+1}{2}\partial \\ &= \cos\left(2m\partial + \frac{\partial}{m}\right) + \lambda\sin\left(2m\partial + \frac{\partial}{m}\right) \\ &= \cos\frac{\partial}{m} + \lambda\sin\frac{\partial}{m}, \end{aligned}$$

therefore the group contains the argument $\frac{\partial}{m}$, which is half of $\frac{2\partial}{m}$, and thus the proposition is proved. With this demonstration ends the possibility of forming any other class of finite quaternion groups than those above enumerated.

Tables of the Five Different Types of Groups.

The explicit formulae which constitute the five different types of groups are contained in the following tables. $N=$ order of the group.

I_m.

The Cyclotomic Groups.

These have the form

$$q^a,\ q^{2a},\ \ldots\ q^{ma};\quad N=m,\ a \text{ prime to } m;$$

where q is any quaternion having the period m.

J_{4n}.

1. *The Double-Pyramid Groups.*

These have the form

$$\begin{array}{l} q^a,\ q^{2a},\ \ldots\ q^{2na}, \\ \mu q^a,\ \mu q^{2a},\ \ldots\ \mu q^{2na};\quad N=4n,\ a \text{ prime to } 2n; \end{array}$$

where q is any quaternion having the period $2n$, and where μ is a vector lying in the plane perpendicular to the axis of q. $[S\lambda\mu=0.]$

2. Q_8: *The i-j-k Group*, viz:

$$\pm 1,\ \pm i,\ \pm j,\ \pm k.$$

This is a special case of J_{4n}.

Q_{24}.

The Double-Tetrahedron Group.

$$\begin{array}{cccc} i^\epsilon, & j^\epsilon, & k^\epsilon, & \\ \left(\frac{1+i+j+k}{2}\right)^\eta, & \left(\frac{1-i-j+k}{2}\right)^\eta, & \left(\frac{1+i-j-k}{2}\right)^\eta, & \left(\frac{1-i+j-k}{2}\right)^\eta; \end{array}$$

$$\epsilon=1,\ 2,\ 3,\ 4;\quad \eta=1,\ 2,\ 3,\ 4,\ 5,\ 6;\quad N=24.$$

Q_{48}.

The Double-Oktahedron Group.

$$\left(\frac{1+i}{\sqrt{2}}\right)^\epsilon,\qquad \left(\frac{1+j}{\sqrt{2}}\right)^\epsilon,\qquad \left(\frac{1+k}{\sqrt{2}}\right)^\epsilon;$$

$$\left(\frac{1+i+j+k}{2}\right)^\eta,\quad \left(\frac{1-i-j+k}{2}\right)^\eta,\quad \left(\frac{1+i-j-k}{2}\right)^\eta,\quad \left(\frac{1-i+j-k}{2}\right)^\eta;$$

$$\begin{array}{ccc} \left(\frac{j+k}{\sqrt{2}}\right)^\zeta, & \left(\frac{k+i}{\sqrt{2}}\right)^\zeta, & \left(\frac{i+j}{\sqrt{2}}\right)^\zeta, \\ \left(\frac{j-k}{\sqrt{2}}\right)^\zeta, & \left(\frac{k-i}{\sqrt{2}}\right)^\zeta, & \left(\frac{i-j}{\sqrt{2}}\right)^\zeta; \end{array}$$

$$\epsilon=1,\ \ldots\ 8;\quad \eta=1,\ \ldots\ 6;\quad \zeta=1,\ \ldots\ 4;\quad N=48.$$

Q_{120}.

1. *The Double-Ikosahedron Group.*

$$\begin{array}{ccc}
i^\epsilon & j^\epsilon & k^\epsilon \\
\left(\frac{i+e'j+ek}{2}\right)^\epsilon, & \left(\frac{j+e'k+ei}{2}\right)^\epsilon, & \left(\frac{k+e'i+ej}{2}\right)^\epsilon, \\
\left(\frac{i-e'j+ek}{2}\right)^\epsilon, & \left(\frac{j-e'k-ei}{2}\right)^\epsilon, & \left(\frac{k+e'i-ej}{2}\right)^\epsilon, \\
\left(\frac{i+e'j-ek}{2}\right)^\epsilon, & \left(\frac{j-e'k+ei}{2}\right)^\epsilon, & \left(\frac{k-e'i-ej}{2}\right)^\epsilon, \\
\left(\frac{i-e'j-ek}{2}\right)^\epsilon, & \left(\frac{j+e'k-ei}{2}\right)^\epsilon, & \left(\frac{k-e'i+ej}{2}\right)^\epsilon;
\end{array}$$

$$\left(\frac{1+i+j+k}{2}\right)^\eta, \quad \left(\frac{1-i-j+k}{2}\right)^\eta, \quad \left(\frac{1+i-j-k}{2}\right)^\eta, \quad \left(\frac{1-i+j-k}{2}\right)^\eta,$$

$$\begin{array}{ccc}
\left(\frac{1+ej+e'k}{2}\right)^\eta, & \left(\frac{1+ek+e'i}{2}\right)^\eta, & \left(\frac{1+ei+e'j}{2}\right)^\eta, \\
\left(\frac{1+ej-e'k}{2}\right)^\eta, & \left(\frac{1+ek-e'i}{2}\right)^\eta, & \left(\frac{1+ei-e'j}{2}\right)^\eta; \\
\left(\frac{e+e'j+k}{2}\right)^\zeta, & \left(\frac{e+e'k+i}{2}\right)^\zeta, & \left(\frac{e+e'i+j}{2}\right)^\zeta, \\
\left(\frac{e+e'j-k}{2}\right)^\zeta, & \left(\frac{e+e'k-i}{2}\right)^\zeta, & \left(\frac{e+e'i-j}{2}\right)^\zeta;
\end{array}$$

$$\varepsilon=1,\ldots.4; \quad \eta=1,\ldots.6; \quad \zeta=1,\ldots.10; \quad N=120.$$

$$e=\frac{1+\sqrt{5}}{2}, \quad e'=\frac{1-\sqrt{5}}{2}.$$

2. Q'_{120}: A second Double-Ikosahedron Group differing from the above by the interchange of e with e'.

The groups Q_8 and Q_{24} are contained in Q_{48}, and Q_8 in Q_{24}, as permutable sub-groups.* Also a double-pyramid group of the $4m^{\text{th}}$ order contains a cyclotomic group of the $2m^{\text{th}}$ order as a *permutable* (ausgezeichnet) sub-group. The group Q_{120} contains no *permutable* sub-group except the group ± 1 of the 2^{d} order.

The vectors of the quaternions of these various groups represent geometrically the axes of symmetry of the corresponding regular polyhedra; that is, they are the diameters of the circumscribed sphere, which pass either through summits, or middle points of faces, or middle points of edges.

* C. Jordan, "Traité des substitutions," pp. 1–50; and W. Dyck, "Gruppe und Irrationalität regulärer Riemann'schen Flächen," *Math. Ann.* Bd. XVII, p. 481.

The Groups of Movements of the Regular Polyhedra in three-dimensional Space.

It is well known that any displacement of a sphere about its centre as a fixed point can be replaced by a simple rotation about a determinate axis, and that such a rotation is represented by the operator $q\,(\;)\,q^{-1}$, q being a quaternion whose axis is the axis of rotation and the double of whose angle is the amount of rotation. From the foregoing discussion, therefore, it follows immediately that all the finite groups of automorphic transformations of the sphere are represented by the various groups of operations of the form $q\,(\;)\,q^{-1}$, when, for a given group, q assumes all the values of the different quaternions in some one of the above enumerated quaternion groups. The linear transformations of Gordan are therefore represented in the following scheme.

$I_m\,(\;)\,I_m^{-1}$. If q assume all the values of the quaternions of a cyclotomic group of the m^{th} or $2m^{th}$ order, m being odd, then the operators of $q\,(\;)\,q^{-1}$, $q^2\,(\;)\,q^{-2}$, ... $q^m\,(\;)\,q^{-m}$ represent the m movements, or rotations about the axis of q, which transport any vertex of a polygon of m sides—whose plane is perpendicular to the axis of q—into all the other vertices of the polygon successively. These rotations I call the automorphic or self-congruent movements of the polygon.*

$J_{4n}\,(\;)\,J_{4n}^{-1}$. If q assume successively the values of the quaternions of a double-pyramid group of the order $4n$, the resulting rotations $q\,(\;)\,q^{-1}$ will represent all the self-congruent movements (*Bewegungen in sich*) of a double pyramid having in all $4n$ faces.

$Q_{24}\,(\;)\,Q_{24}^{-1}$. If q assume successively the values of the quaternions of the group Q_{24}, then the operators $q\,(\;)\,q^{-1}$ represent the 12 self-congruent movements of the tetrahedron, such that its four summits

$$\frac{i+j+k}{\sqrt{3}}, \qquad \frac{-i-j+k}{\sqrt{3}}, \qquad \frac{i-j-k}{\sqrt{3}}, \qquad \frac{-i+j-k}{\sqrt{3}},$$

its four centres of faces

$$-\frac{i+j+k}{\sqrt{3}}, \qquad -\frac{-i-j+k}{\sqrt{3}}, \qquad -\frac{i-j-k}{\sqrt{3}}, \qquad -\frac{-i+j-k}{\sqrt{3}},$$

and its six middle points of edges

$$\pm i, \qquad \pm j, \qquad \pm k,$$

are separately permuted with each other.

*I call a figure self-congruent, when without undergoing any change as a whole, like parts—as points with points, lines with lines, etc.—are interchanged. Self-congruent movements or rotations are those which produce such an interchange of like elements; thus all the motions of a sphere about its centre as a fixed point are self-congruent movements.

$Q_{48}(\ )Q_{48}^{-1}$. If the quaternions of the group Q_{48} be substituted for q in $q(\ )q^{-1}$, the resulting transformations will be the 24 self-congruent movements of the oktahedron. The summits of this oktahedron are

$$\pm i, \qquad \pm j, \qquad \pm k;$$

its centres of faces are

$$\pm\frac{i+j+k}{\sqrt{3}}, \quad \pm\frac{-i-j+k}{\sqrt{3}}, \quad \pm\frac{i-j-k}{\sqrt{3}}, \quad \pm\frac{-i+j-k}{\sqrt{3}};$$

and its middle points of edges are

$$\pm\frac{j\pm k}{\sqrt{2}}, \quad \pm\frac{k\pm i}{\sqrt{2}}, \quad \pm\frac{i\pm j}{\sqrt{2}};$$

and the movements in question permute these *like* elements with each other.

$Q_{120}(\ )Q_{120}^{-1}$. Finally, if in $q(\ )q^{-1}$, q be made to assume all the values in Q_{120}, the result will be the 60 self-congruent movements of the ikosahedron. We have here for summits

$$\pm\frac{e'j\pm k}{e'\sqrt{5}}, \quad \pm\frac{e'k\pm i}{e'\sqrt{5}}, \quad \pm\frac{e'i\pm j}{e'\sqrt{5}};$$

for centres of faces

$$\pm\frac{i+j+k}{\sqrt{3}}, \quad \pm\frac{-i-j+k}{\sqrt{3}}, \quad \pm\frac{i-j-k}{\sqrt{3}}, \quad \pm\frac{-i+j-k}{\sqrt{3}},$$

$$\pm\frac{ej\pm e'k}{\sqrt{3}}, \quad \pm\frac{ek\pm e'i}{\sqrt{3}}, \quad \pm\frac{ei\pm e'j}{\sqrt{3}};$$

and for middle points of edges

$$\pm i, \qquad \pm j, \qquad \pm k,$$

$$\pm\frac{i\pm e'j\pm ek}{2}, \quad \pm\frac{j\pm e'k\pm ei}{2}, \quad \pm\frac{k\pm e'i\pm ej}{2}.$$

The above scheme contains all the possible finite groups of automorphic linear transformations of the sphere, or what is the same thing, of the plane of complex variables.

SCHWARZBACH, SAXONY, *September*, 1881.